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Very stable extensions on arithmetic surfaces

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Abstract Given a line bundle L on a smooth projective curve over the complex numbers, we show that a general extension E of L by the trivial line bundle is *very stable*: line bundles contained in E have degree much less than half the degree of E. From this result we deduce new inequalities for the successive minima of the euclidean lattice $H^1(X, L^{-1})$, where L is an hermitian line bundle on the arithmetic surface X.

Keywords Projective curve · Semi-stable bundle · Secant variety · Arithmetic surface · Successive minima

Mathematics Subject Classification (2000) MSC 14H60 · MSC 14G40

1 Introduction

Let X be an arithmetic surface and $\bar{N}$ an hermitian line bundle on X. The lattice

$$\Lambda = H^1(X, N^{-1})$$

is equipped with the L^2-metric. In this paper we keep on studying the successive minima of this euclidean lattice; see [2], [3] and [4] for previous results. When the degree of N is large enough we get a lower bound for the k-th minimum of Λ, when $k > \frac{\deg(N)}{2} + g$, where g is the generic genus of X; cf. Theorem 2 for a precise statement.

As in *op. cit.*, we get this inequality by considering the extension

$$0 \to \mathcal{O}_X \to E \to N \to 0$$

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defined by a class $e \in \Lambda$. If $a \geq 0$ is an integer, we say that e is *a-stable* when the restriction of E to the geometric generic fiber C of X does not contain any line bundle L with

$$\deg(L) > \frac{\deg(E) - a}{2} \, .$$

The main ingredient in the proof of Theorem 2 is the assertion that any $V \subset H^1(C, N^{-1})$ contains a class e which is a-stable when $\dim(V)$ is large enough (Theorem 1). This is proved by induction, the case $a = 0$ being Proposition 2 in [4].

The paper is organized as follows. In Section 1 we introduce the notion of a-stability for a rank two vector bundle on C. The Lemma 1 relates a-stability and semi-stability when E is an extension of line bundles. In Lemma 2 we introduce secant varieties. Sections 1.4 to 1.9 are then devoted to the proof of Theorem 1. In Section 2 we let $\bar{N}$ be an hermitian line bundle on some arithmetic surface X. Proposition 2 gives a lower bound for the L^2- norm of $e \in \Lambda$ if its restriction to C is a-stable. Theorem 2 follows by arguments similar to those in [2], [3] and [4].

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2 Very stable extensions on curves

2.1

Let k be an algebraically closed field of characteristic zero and C a smooth projective curve of genus g over k. Let $a \geq 0$ be an integer. A rank two vector bundle E over C is said to be *a-stable* when, for every line bundle L contained in E, the following inequality holds:

$$\deg(L) \leq \frac{\deg(E) - a}{2} \, .$$

So, E is semi-stable (resp. stable) iff it is 0-stable (resp. 1-stable).

2.2

Let M and L be two line bundles on C and

$$0 \to L \to E \to M \to 0$$

an extension of M by L. Let A be an effective line bundle of degree a on C and $s : \mathcal{O}_C \to A$ a non trivial global section of A on C. If A^{-1} is the dual of A and MA^{-1} its tensor product with M, the section s defines an injective morphism

$$i : MA^{-1} \to M\,.$$

If we pull-back the extension E by i we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle i} & & \\
0 & \longrightarrow & L & \longrightarrow & E' & \longrightarrow & MA^{-1} & \longrightarrow & 0
\end{array}$$

for some rank two vector bundle E' on C.

Lemma 1. *If E is a-stable, E' is semi-stable.*

Proof. The morphism $E' \to E$ is injective, therefore any line bundle N contained in E' is also contained in E. Hence

$$\deg(N) \leq \frac{\deg(E) - a}{2} = \frac{\deg(E')}{2}$$

and E' is semi-stable.

2.3

Let N be a line bundle of degree $n \geq 3$ on C. Each cohomology class

$$e \in H^1(C, N^{-1}) = \mathrm{Ext}(N, \mathcal{O}_C)$$

classifies an extension

$$0 \to \mathcal{O}_C \to E \to N \to 0$$

of N by the trivial line bundle. We say that e is *a-stable* (resp. *semi-stable*) if E is a-stable (resp. semi-stable).

Let

$$\mathbb{P} = \mathbb{P}(H^1(C, N^{-1}))$$

be the projective space of lines in $H^1(C, N^{-1})$. If ω is the sheaf of differentials on C, Serre duality implies that $H^1(C, N^{-1}) \simeq H^0(C, \omega \otimes N)^*$ and we get a canonical immersion $C \to \mathbb{P}$. If D is an effective divisor on C we let $\langle D \rangle \subset \mathbb{P}$ be the linear span of D, and $|D|$ be the support of D. For every integer $d \geq 0$ we consider the secant variety

$$\Sigma_d = \bigcup_{\deg(D)=d} \langle D \rangle\,.$$

Lemma 2. *The extension class e is a-stable iff its image $\bar{e}$ in $\mathbb{P}$ does not belong to Σ_d when $d < \frac{n+a}{2}$.*

Proof. This follows from the arguments discussed in [1] p. 451, [3] §1.6 or [4] §2.4.2.

2.4

We keep the notation of the previous paragraph.

Theorem 1. *Assume that $n \geq a+3$ and let $V \subset H^1(C, N^{-1})$ be a k-vector space of dimension*
$$\dim(V) \geq \frac{n+a}{2} + g\,. \tag{1}$$
Then there exists a class $e \in V$ which is a-stable.

In view of Lemma 2, Theorem 1 can be rephrased as follows. Let $\delta = (n+a)/2$. Assume that $n \geq \delta + 2$. When $d < \delta$ the secant variety Σ_d does not contain any linear subspace $\mathbb{P}(V)$ with $\dim(V) \geq \delta + g$.

2.5

To prove Theorem 1 we can assume that $n+a$ is even. Indeed, if $n+a$ is odd the condition (1) is equivalent to

$$\dim(V) \geq \frac{n+a+1}{2} + g\,,$$

and, if e is $(a+1)$-stable, it is also a-stable.

When $n+a$ is even, we proceed by induction on a. When $a=0$ (and n is even) Theorem 1 is Proposition 2 in [4].

Assume Theorem 1 has been proved for $a-1$. If $P \in C(k)$ is a point on C we let
$$X_P = \bigcup_{\substack{P \in |D| \\ \deg(D) < \frac{n+a}{2}}} \langle D \rangle\,,$$
and we consider a linear subspace $V \subset H^1(C, N^{-1})$ of dimension at least $\frac{n+a}{2} + g$. Assume that P does not lie in the projective space $\mathbb{P}(V) \subset \mathbb{P}$.

Lemma 3. *The intersection $X_P \cap \mathbb{P}(V)$ is a proper closed subset of $\mathbb{P}(V)$.*

2.6

To prove Lemma 3, let $N^{-1}P$ be the tensor product of N^{-1} with the line bundle $\mathcal{O}(P)$ and
$$\pi : H^1(C, N^{-1}) \to H^1(C, N^{-1}P)$$
the corestriction morphism. Let

$$\mathbb{P}' = \mathbb{P}(H^1(C, N^{-1}P))$$

and let
$$p : \mathbb{P} - \{P\} \to \mathbb{P}'$$

be the linear projection defined by π. Since P is not in $\mathbb{P}(V)$, we have $\pi(V) = V'$, where V' has the same dimension as V, and p induces an isomorphism

$$\mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V')\,.$$

If D is a divisor on C such that $P \in |D|$, $p(\langle D\rangle)$ is the linear span $\langle D-P\rangle'$ of $D-P$ in $\mathbb{P}'$. The secant variety

$$\Sigma = \bigcup_{\deg(D)<\frac{n+a}{2}-1} \langle D\rangle'$$

is a closed subset of $\mathbb{P}'$, hence its inverse image

$$X_P - \{P\} = p^{-1}(\Sigma)$$

is a closed subset of $\mathbb{P} - \{P\}$.

If $\mathbb{P}(V)$ was contained in X_P, $\mathbb{P}(V')$ would be contained in Σ. But

$$\dim(V') = \dim(V) \geq \frac{n+a}{2} + g > \frac{(n-1)+(a-1)}{2} + g$$

hence, by the induction hypothesis, $\mathbb{P}(V')$ contains a point $\bar{e}'$ such that e' is $(a-1)$-stable. Since

$$\frac{n+a}{2} - 1 = \frac{(n-1)+(a-1)}{2}\,,$$

$\bar{e}'$ does not lie in Σ (Lemma 2). This proves Lemma 3.

2.7

To prove Theorem 1 we can assume that $\dim(V) = \frac{n+a}{2} + g$. Since $H^1(C, N^{-1})$ has dimension $n+g-1$ and $n \geq 3$, V is a proper subspace of $H^1(C, N^{-1})$, and $\mathbb{P}(V)$ does not contain C. Let $P_1, \ldots, P_a$ be a distinct points of $C \backslash \mathbb{P}(V)$ and A the divisor

$$A = P_1 + \cdots + P_a\,.$$

¿From Lemma 3 we conclude that

$$U = \mathbb{P}(V) - \bigcup_{\substack{|A|\cap|D|\neq\emptyset \\ \deg(D)<\frac{n+a}{2}}} \langle D\rangle$$

is a nonempty open subset of $\mathbb{P}(V)$. Let $N^{-1}A^{-1}$ be the tensor product of N^{-1} with $\mathcal{O}(-A)$ and

$$\pi : H^1(C, N^{-1}A^{-1}) \to H^1(C, N^{-1})$$

the corestriction map. Let $\mathbb{P}' = \mathbb{P}(H^1(C, N^{-1}A^{-1}))$ and

$$p : \mathbb{P}' - \langle A\rangle' \to \mathbb{P}$$

the projection induced by π.

By Proposition 1 below, applied to NA instead of N and to $W = \pi^{-1}(V)$, there exists a non trivial class $e \in V$ such that $\bar{e} \in U$ and each $e' \in H^1(C, N^{-1} A^{-1})$ such that $\pi(e') = e$ is semi-stable. Assume $\bar{e}$ lies in $\langle D \rangle$, for some effective divisor D on C. Then, either $\deg(D) \geq \frac{n+a}{2}$ or $|A| \cap |D| = \emptyset$ and $\deg(D) < \frac{n+a}{2}$. In the latter case, since

$$\deg(NA\,\omega) = (2g-2) + n + a > 2g - 2 + \deg(A) + \deg(D)\,,$$

we have

$$\langle A \rangle \cap \langle D \rangle = \langle A \cap D \rangle = \emptyset$$

([1] p. 434) and there exists $\bar{e}' \in \langle D \rangle'$ such that $p(\bar{e}') = \bar{e}$. Since e' is semi-stable and $\deg(NA) = n + a$, Lemma 2 implies that

$$\deg(D) \geq \frac{n+a}{2}\,.$$

Applying Lemma 2 again, we conclude that e is a-stable.

2.8

Let N be a line bundle of even positive degree n on C. Let

$$K \subset W \subset H^1(C, N^{-1})$$

be linear subspaces. We assume that $V = W/K$ is not zero and we let $U \subset \mathbb{P}(V)$ be a nonempty open subset. Let $\pi : W \to V$ be the projection and $a = \dim(K)$.

Proposition 1. *If $\dim(V) \geq \frac{n}{2} + g$ there exists $\varepsilon \in V$ such that $\bar{\varepsilon} \in U$ and any $e \in W$ such that $\pi(e) = \varepsilon$ is semi-stable.*

2.9

To prove Proposition 1, we first note, as in [4] p. 288, that there exist two line bundles L and M on C such that $LM = \omega$ and $ML^{-1} = N$. Any class $e \in H^1(C, N^{-1})$ defines an extension

$$0 \to L \to E \to M \to 0$$

and a boundary map

$$\partial_e : H^0(C, M) \to H^1(C, L)\,.$$

The bundle E is semi-stable iff ∂_e is an isomorphism. We now adapt to our situation the argument of C. Voisin in [4] 2.2. Let

$$\mu : H^0(C, M)^{\otimes 2} \to W^*$$

be the composite of the cup-product with the projection

$$H^0(C, M^2) = H^1(C, N^{-1})^* \to W^* .$$

Any vector $e \in W$ defines, via μ, a quadric q_e in the projective space $\mathbb{P}(H^0(C, M))$. The boundary map ∂_e is an isomorphism iff q_e is non singular.

Arguing by contradiction, we assume that, for every $\varepsilon \in V$ such that $\bar{\varepsilon} \in U$, there exists $e \in W$ such that $\pi(e) = \varepsilon$ and q_e is singular. When $r \geq 1$ is a positive integer, we let $U_r \subset U$ be the set of those $\bar{\varepsilon}$ such that there exist $e \in W$ with $\pi(e) = \varepsilon$ and the singular locus of q_e has dimension r. We have

$$U = \bigcup_{r \geq 1} U_r$$

and each set U_r is constructible. Therefore there exists r_0 such that U_{r_0} contains a dense open subset of $\mathbb{P}(V)$. Consider the Zariski closure $B \subset \mathbb{P}(H^0(C, M))$ of the union of the singular loci of the quadrics with singular locus of dimension r_0, and let b be the dimension of B.

Let $\sigma \in H^0(C, M)$ be a representative of a generic point $\bar{\sigma} \in B$. We claim that the map

$$\mu_\sigma : H^0(C, M) \to W^*$$

sending τ to $\mu(\sigma \otimes \tau)$ has rank at most $a + b$. Indeed $q \in W$ is singular at τ iff it lies in the subspace $Q_\tau \subset W$ orthogonal to the image of μ_τ. The union of all the vector spaces Q_τ, $\bar{\tau} \in B$, maps onto U_{r_0}. Therefore the dimension of Q_σ is at least $\dim(V) - b$ and the rank of μ_σ is at most $\dim(W) - (\dim(V) - b) = a + b$, as claimed.

It follows that the kernel H_σ of μ_σ has dimension $c \geq m - a - b$, where $m = \dim H^0(C, M)$. Arguing as in *op. cit.*, p. 290, we find that the subspace $W^\perp \subset H^0(C, M^2)$ orthogonal to W has dimension at least

$$b + c \geq m - a .$$

Therefore, since $H^1(C, N^{-1})$ has dimension $n + g - 1$, W has dimension at most $n + a + g - m - 1$. By Riemann-Roch and the fact that $2\deg(M) = 2g - 2 + n$, we know that

$$n - m + g \leq \frac{n}{2} + g .$$

Since $\dim(V) = \dim(W) - a$, we get

$$\dim(V) \leq \frac{n}{2} + g - 1 ,$$

contradicting our hypothesis.

3 Arithmetic surfaces

3.1

Let F be a number field, $\mathcal{O}_F$ its ring of integers and $S = \mathrm{Spec}(\mathcal{O}_F)$. Consider a proper flat curve X over S such that X is regular and the generic fiber X_F is geometrically irreducible of genus g. Let

$$\deg : \mathrm{Pic}(X) \to \mathbb{Z}$$

be the morphism which sends the class of a line bundle over X to the degree of its restriction to X_F.

Let $\bar{N} = (N, h)$ be an hermitian line bundle on X. The cohomology group

$$\Lambda = H^1(X, N^{-1})$$

is a finitely generated module over $\mathcal{O}_F$. It can be endowed as follows with an hermitian norm. For every complex embedding $\sigma : F \to \mathbb{C}$, we let $X_\sigma = X \underset{\mathcal{O}_F}{\otimes} \mathbb{C}$ be the corresponding complex curve. The cohomology group

$$\Lambda_\sigma = \Lambda \otimes \mathbb{C} = H^1(X_\sigma, N_{\mathbb{C}}^{-1})$$

is canonically isomorphic to the complex vector space $\mathcal{H}^{01}(X_\sigma, N_{\mathbb{C}}^{-1})$ of harmonic differential forms of type $(0,1)$ with coefficients in the restriction $N_{\mathbb{C}}^{-1}$ of N^{-1} to X_σ. Given $\alpha \in \mathcal{H}^{01}(X_\sigma, N_{\mathbb{C}}^{-1})$ we let α^* be its transposed conjugate (the definition of which involves h), and we define

$$\|\alpha\|_{L^2}^2 = \frac{i}{2\pi} \int_{X_\sigma} \alpha^* \alpha \, .$$

Given $e \in \Lambda$ we let

$$\|e\| = \operatorname*{Sup}_{\sigma} \|\sigma(e)\|_{L^2} \, ,$$

where σ runs over all complex embeddings of F.

Let $a \geq 0$ be an integer and n the degree of N. We assume that $n \geq a + 3$. Let $\bar{A}$ be an hermitian line bundle on X of degree $\deg(A) = a$, and $s : \mathcal{O}_X \to A$ a non zero global section of A. Define

$$\|s\|_{\sup} = \operatorname*{Sup}_{x \in X(\mathbb{C})} \|s(x)\| \, ,$$

where $X(\mathbb{C}) = \coprod_\sigma X_\sigma$ is the set of complex points of X.

Any class $e \in \Lambda$ defines an extension

$$0 \to \mathcal{O}_X \to E \to N \to 0$$

on X. If $\bar{F}$ is a fixed algebraic closure of F, we let $E_{\bar{F}}$ be the restriction of E to $X_F \otimes \bar{F}$. Denote by $r = [F : \mathbb{Q}]$ the absolute degree of F.

Proposition 2. *Assume $E_{\bar{F}}$ is a-stable. Then the following inequality holds*

$$\log \|e\| \geq \frac{(\bar{N}-\bar{A})^2}{2(n-a)\,r} - \log \|s\|_{\sup} - 1\,,$$

where $(\bar{N}-\bar{A})^2 \in \mathbb{R}$ denotes the arithmetic self-intersection of the first Chern class $\hat{c}_1(\bar{N}\bar{A}^{-1}) \in \widehat{CH}^1(X)$.

3.2

To prove Proposition 2 we consider the extension

$$0 \to \mathcal{O}_X \to E' \to NA^{-1} \to 0$$

obtained by pulling back $e \in H^1(X, N^{-1})$ to $e' \in H^1(X, N^{-1}A)$. Since the restriction of E' to $X_{\bar{F}}$ is semi-stable (Lemma 1) we have

$$\log \|e'\| \geq \frac{(\bar{N}-\bar{A})^2}{2(n-a)\,r} - 1 \tag{2}$$

(see [2] or [4] pp. 294-295). So we are left with comparing $\|e\|$ and $\|e'\|$.

We have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & E' & \longrightarrow & NA^{-1} & \longrightarrow & 0\,.
\end{array}$$

Any C^∞ splitting $E_{\mathbb{C}} \to \mathbb{C}$ of the top extension defines, by restriction, a C^∞ splitting $E'_{\mathbb{C}} \to \mathbb{C}$. The Cauchy-Riemann operators $\bar{\partial}_E$ and $\bar{\partial}_{E'}$ can then be written as matrices

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{\mathbb{C}} & \alpha \\ 0 & \bar{\partial}_N \end{pmatrix}$$

and

$$\bar{\partial}_{E'} = \begin{pmatrix} \bar{\partial}_{\mathbb{C}} & \alpha' \\ 0 & \bar{\partial}_{NA^{-1}} \end{pmatrix}\,,$$

where α is a linear map $C^\infty(N_{\mathbb{C}}) \to A^{01}(\mathbb{C})$, and $\alpha' : C^\infty(NA_{\mathbb{C}}^{-1}) \to A^{01}(\mathbb{C})$ is the restriction of α to $NA_{\mathbb{C}}^{-1}$.

For any $\sigma : F \to \mathbb{C}$, choose a local chart z of X_σ and local trivializations of $N_{\mathbb{C}}$ and $A_{\mathbb{C}}$. We have

$$\alpha = \varphi\, d\bar{z}\,,$$

where φ is a smooth function and

$$\alpha' = \varphi u\, d\bar{z}\,,$$

where u is the local section of A defined by s. The transposed conjugates are

$$\alpha^* = \frac{\bar{\varphi}}{h_N(1,1)}\, dz$$

and

$$\alpha'^* = \frac{h_A(1,1)\,\bar{\varphi}\,\bar{u}\,dz}{h_N(1,1)}\,,$$

where $h_N(1,1)$ (resp. $h_A(1,1)$) is the squared norm of the local generator of N (resp. A). It follows that

$$\alpha'^*\,\alpha' = h_A(1,1) u\,\bar{u}\,\alpha^*\alpha = \|s\|^2\,\alpha^*\alpha\,,$$

and

$$\|\alpha'\|^2_{L^2} = \frac{i}{2\pi'}\int_{X_\sigma} \alpha'^*\,\alpha' \leq \|s\|^2_{\sup}\,\|\alpha\|^2_{L^2}\,.$$

Assume that the splitting $E_{\mathbb{C}} \to \mathbb{C}$ has been chosen such that α is harmonic. Then we get

$$\|\alpha'\|_{L^2} \leq \|s\|_{\sup}\,\|\sigma(e)\|_{L^2}\,.$$

Since $\|\sigma(e')\|_{L^2}$ is the smallest value of $\|\alpha'\|_{L^2}$ when α' runs over all representatives of e' in $A^{01}(X_\sigma, N^{-1}A_{\mathbb{C}})$, we get

$$\|\sigma(e')\|_{L^2} \leq \|s\|_{\sup}\,\|\sigma(e)\|_{L^2}$$

hence

$$\|e'\| \leq \|s\|_{\sup}\,\|e\|\,.$$

This inequality and (2) imply Proposition 2.

3.3

We keep the notation of §2.1 and we consider the (logarithms of the) successive minima of the euclidean lattice $(\mathit{\Lambda}, \|\cdot\|)$. When $k \leq rk(\mathit{\Lambda})$, μ_k is the infimum of all real numbers μ such that there exists k elements $e_1, \ldots, e_k$ in $\mathit{\Lambda}$ which are linearly independent in $\mathit{\Lambda} \otimes F$ and such that

$$\|e_i\| \leq \exp(\mu) \qquad \text{for all} \quad i = 1, \ldots, k\,.$$

Theorem 2. *Assume that*

$$\frac{n+a}{2} + g \leq k < n + g - 1\,.$$

Then

$$\mu_k \geq \frac{(\bar{N} - \bar{A})^2}{2(n-a)\,r} - \log\|s\|_{\sup} - C\,,$$

where $C = 1 + \log(d(n,a)\,k)$*, and* $d(n,a)$ *is bounded as in (3) below.*

3.4

To prove Theorem 2 we let

$$V \subset H^1(X_{\bar F}, N^{-1}) = \Lambda \otimes \bar F$$

be the linear space spanned by $e_1, \dots, e_k$. Since $k < n + g - 1$, V is a proper subspace of $\Lambda \otimes \bar F$. From Theorem 1 we know that there exists $e \in V$ such that the corresponding extension E of N by $\mathcal{O}_C$ on $C = X_{\bar F}$ is a-stable. More precisely E is a-stable when $\bar e$ does not belong to $\mathbb{P}(V) \cap H(n,a)$, where $H(n,a)$ is an hypersurface defined as follows. When $n + a$ is odd we let $H(n,a) = H(n, a+1)$. When $n + a$ is even, $H(n,a)$ is defined by induction on a. We choose $A = P_1 + \ldots + P_a$ as in 1.7. The class $\bar e$ is a-stable when it satisfies the following two conditions. First, for any $P \in |A|$, the projection of $\bar e$ into $\mathbb{P}(H^1(C, N^{-1}P))$ should not lie in $H(n-1, a-1)$. Second, let L and M be line bundles on C such that $LM = \omega$ and $ML^{-1} = NA$; then, any class $\bar e' \in \mathbb{P}(H^1(C, N^{-1}A^{-1}))$ which maps to $\bar e \in \mathbb{P}(H^1(C, N^{-1}))$ should be such that the boundary map

$$\partial_{e'} : H^0(C, M) \to H^1(C, L)$$

is an isomorphism. Let m be the dimension of $H^0(C,M)$, $\sigma_1, \dots, \sigma_m$ a basis of $H^0(C,M)$, and $\tau_1, \dots, \tau_m$ a basis of $H^1(C,L)^*$. Then $\partial_{e'}$ is injective as soon as it satisfies the inequation

$$(\partial_{e'}(\sigma_1) \wedge \ldots \wedge \partial_{e'}(\sigma_m), \tau_1 \wedge \ldots \wedge \tau_m) \neq 0\,,$$

which is of degree $m \leq \frac{n+a}{2}$ in e'. It follows from the proof of Theorem 1 that $\bar e$ is a-stable as soon as it satisfies these two conditions, which is the case when $\bar e \notin H(n,a)$, where $H(n,a)$ is an hypersurface of degree $d(n,a)$ with

$$d(n,a) \leq \frac{n+a}{2} + a\, d(n-1, a-1)$$

and

$$d(n,0) \leq \frac{n}{2}\,.$$

Therefore we get

$$\begin{aligned} d(n,a) &\leq p + a(p-1) + a(a-1)(p-2) + a(a-1)(a-2)(p-3) \\ &\quad + \ldots + a!(p-a)\,, \qquad \text{when } n + a = 2p \text{ or } 2p - 1\,. \end{aligned} \tag{3}$$

Therefore, as in [3] Prop. 5, there exist k integers $n_1, \dots, n_k$, with $|n_i| \leq d(n,a)$ for all i, such that

$$e = n_1\, e_1 + \ldots + n_k\, e_k$$

does not lie in $H(n,a)$. The extension E defined by e on X is then a-stable, and Proposition 2 implies that

$$\log \|e\| \geq \frac{(\bar N - \bar A)^2}{2(n-a)\, r} - \log \|s\|_{\sup} - 1\,.$$

Since

$$\|e\| \leq k\, d(n,a)\, \exp(\mu_k)\,,$$

Theorem 2 follows.

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